Will a System Operate?

Cindy Yerges

Abstract: *A system of n objects are lined in a row. There are m defective objects in this system. The system will be operable if no more than k defective objects are adjacent in this line. Using restricted equations and inclusion-exclusion, we will compute the probability that a randomly selected system of n objects with m defective objects will be operable.*

Introduction

A system of antennae consists of defective and functioning antennae. In a system antennae, the system will be operable if no more than k defective antennae are adjacent. If we choose a system of n antennae with m defectives from a uniform set of all configurations, we can calculate the probability of selecting an operable system using restricted equations and inclusion-exclusion.

Analysis

Theorem 1

The probability of selecting an operable system is

$$\sum_{i=0}^{n/(k+1)} \frac{\binom{n+m-1}{i}\binom{n-(k+1)i}{n-m}(-1)^i}{\binom{n}{m}}$$

assuming that $\binom{a}{b} = 0$ for b > a.

Proof of Theorem 1

The total number of configurations of systems of n antennae with m defectives is $\binom{n}{m}$. Our task is to count the number of operable configurations.

A system consists of m defective and n-m functioning antennae. With respect to the functioning antennae, there are (n-m)+1 positions to place defective antennae -- one next to each functioning antennae, and one at the end of the line. Each position can hold between 0 and k defective antennae and the system will still be operable.

Therefore, letting O's denote operable antennae, we can represent a system of antennae as

$$\underset{x_1}{\_}\quad O\quad \underset{x_2}{\_}\quad O\quad \underset{x_3}{\_}\quad \ldots\quad O\quad \underset{x_{n-m+1}}{\_}$$

Figure 1: system of operable antannae

where $x_1 + x_2 + x_3 + \ldots + x_{n-m+1} = m$. The number of operable configurations is equal to the number of non-negative solutions to this equation in which each of the x's are at most k.

Let fi(k) equal the number of non-negative solutions to the equation $x_1 + x_2 + x_3 + \ldots + x_{n-m+1} = m$ where i of the x's are greater than k. So, the total number of non-negative solutions to $x_1 + x_2 + x_3 + \ldots + x_{n-m+1} = m$, where each x may hold between 0 and k defective antennae, can be expressed by inclusion-exclusion as

$$f_0(k) - f_1(k) + f_2(k) - \ldots \pm f_{n-m+1}(k).$$

There are $\binom{n-m+1}{i}$ ways to choose the i x's $\geq$ k+1. So,

$$f_i(k) = \binom{n-m+1}{i} \times \begin{bmatrix} \text{the number of non-negative} \\ \text{solutions where} \\ x_1, x_2, \ldots, x_i \geq (k+1) \end{bmatrix}.$$

We can rewrite this as

$$(x_1 - k) + (x_2 - k) + \ldots + (x_i - k) + (x_{i+1} + 1) + \ldots + (x_{n-m+1} + 1) = m - i \cdot k + (n - m + 1 - i).$$

If we let $y_1 = x_1 - k, y_2 = x_2 - k, \ldots y_i = x_i - k, y_{i+1} = x_{i+1} + 1$, we can write the last equation as $y_1 + y_2 + \ldots + y_i + y_{i+1} + \ldots + y_{n-m+1}$ where $y_1 + y_2 + \ldots + y_i + y_{i+1} + \ldots + y_{n-m+1}$. Using the theory of restricted equations [1], the number of solutions to this equation is:

$$\binom{n - i(k + 1)}{n - m}.$$

Therefore, using the equations derived above,

$$f_i(k) = \binom{n - m + 1}{i}\binom{n - i(k + 1)}{n - m}.$$

Thus the probability of selecting an operable system of antennae is as stated in the theorem.

EOP

If k=1, the system will be operable if no defective antennae are present in the system. This case is worked out in [2].

Directions for Further Research

We can ask how the above equation will be affected if the system is circular instead of linear. Or, what if k functioning antennae must be adjacent for the system to be operable, with no regard to the number of adjacent defectives?

Bibliography

[1] Niven, Ivan. *Mathematics by Choice: How to Count Without Counting.* Random House, 1965.

[2] Ross, Sheldon. *A First Course in Probability.* Macmillan Publishing Company, 1988.