Trajectory Modeling of Voyager 2 Under Gravitational Influence of Jupiter

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(Dated: December 13, 2005)

Through a series of differential equations, an accurate model of Voyager 2's trajectory can be made as the probe experiences the powerful gravitational field of Jupiter. By defining the relationship between the Sun, Jupiter, and Voyager 2 upon a modified celestial coordinate plane, both a trajectory map and a plot of probe velocity can be garnered. As suspected, the 'slingshot' experienced by Voyager 2 increases its total energy and bends its path in order to continue its planetary mission. By using actual data from NASA and the Goddard Space Flight Center, a comparison between this model and real-world data is made, showing that the model approximates the actual trajectory well. Discrepancies arise only from the approximations used to initially create the model.

I. INTRODUCTION

Since the beginning of human history, we have looked skyward with both amazement and great interest. The glistening stars, the flash of a meteor, and the procession of the planets have intrigued both scientists and children alike. Great thinkers and innovators such as Galileo, Copernicus, Brache, and Kepler devoted their lives to uncovering the mysteries outside our earthly realm, and modern technology has vastly increased our current knowledge of the solar system and beyond. Perhaps the greatest period of planetary discovery came in the 1970s and 1980s when the United States launched four probes on a tour of the outer solar system. Pioneer 10, Pioneer 11, Voyager 1, and Voyager 2 exponentially expanded our understanding of the solar system, and Voyager 2 may be considered the most important probe of all.

Taking advantage of a rare planetary alignment, Voyager 2 was able to intercept every planet in the outer solar system, sans Pluto. This "Grand Tour" provided a wealth of information and thousands of photographs of the gas giants Jupiter, Saturn, Uranus, and Neptune. Even with the chance planetary alignment, Voyager 2 relied on an immense amount of planning and coordination to successfully complete its mission. With the help of engineers, physicists, and mathematicians, Voyager 2 was launched on a trajectory that used each passing planet as a "slingshot" by absorbing energy from each planet's gravitational field and orbital motion. The gravitational slingshot sent Voyager 2 hurtling towards its next planetary destination, where an additional boost kept Voyager heading outward from the sun.

Using the gravitational slingshot required a great deal of precise calculations, for the consequences are no less than the loss of a 722 kilogram instrument costing 250 million dollars. We can model the effects of this "slingshot" through a series of differential equations, taking into account both the gravity of the sun and the target planet. The differential equations will model the slingshot around Jupiter, the largest and strongest planet encountered by Voyager 2. By tracking both Voyager and Jupiter as they move in their orbits, we can determine the energy absorbed by Voyager as it passes Jupiter. We can also determine the angle of gravitational deflection as the spacecraft overtakes Jupiter on its new trajectory towards Saturn.

II. MODEL FORMATION

Relating the motion and trajectory of Jupiter and Voyager 2 comes through the following differential equations:

$$x'' = \frac{-x}{r^3} - \frac{m_j(x - x_j)}{D^3} \tag{1}$$

$$y'' = \frac{-y}{r^3} - \frac{m_j(y - y_j)}{D^3}. \tag{2}$$

In this situation, r stands for the distance between Voyager and the Sun, and D stands for the distance between Voyager and Jupiter. D and r can be further defined as:

$$r = \sqrt{x^2 + y^2} \tag{3}$$

$$D = \sqrt{(x - x_j)^2 + (y - y_j)^2}. \tag{4}$$

Throughout Eqns. 1-4, Voyager 2 and Jupiter have been set on the same coordinate plane in which the sun acts as the origin at (0, 0). Voyager 2 is at the coordinates (x, y), while Jupiter is at the coordinates (x_j, y_j). To establish this coordinate plane and simplify our model, the multi-dimensional solar ecliptic must be condensed into two dimensions. While both Voyager and Jupiter do not travel in a level orbit around the Sun, their solar latitudes are minimal in comparison to the vast radial distances across the solar system, and thus can be disregarded. (Appendix I)

The elimination of solar latitude places Voyager and Jupiter in a two dimensional plane, from which we can determine initial positions of Jupiter and Voyager.

To fully evaluate the effects on Voyager while inside Jupiter's gravitational field, the model is based upon the time period from May 1, 1979, to September 15, 1979.

Initial conditions can be determined by using daily heliocentric coordinates for Jupiter and Voyager 2 provided by the Goddard Space Flight Center and NASA. These coordinates come in sun-centered latitude and longitude components, along with solar radii in astronomical units (1 AU = 149,598,000 km). The solar-ecliptic longitude is the angle between the given object and an x-axis sent into the constellation Aries during the Vernal Equinox. Because the model looks only upon the closed sun-Jupiter-Voyager 2 system and not upon their relation to the outside universe, we can set our x-axis along an arbitrary line. From this axis, we input angles of longitude and solar radii and use simple trigonometry to determine the initial positions of our interacting bodies (Appendix 1)

In addition to initial positions, an initial velocity of Voyager 2 at the start time, May 1, 1979, is needed in order to evaluate the system of modeling equations. Comparing NASA data and mathematical graphs of Voyager's solar system escape velocity, an estimate of 10 km/s can be established for the time when the spacecraft entered Jupiter's gravitational field. With both initial velocity and initial position, step-by-step evaluation of Voyager's position can be made through Mathematica and a Runge-Kutta numerical approximation for a system of four equations.

III. MODEL PREPARATION

To correctly model the trajectory of Voyager 2, the Mathematica program containing the Runge-Kutta approximation must be properly formatted to coincide with the given real-world data. Most importantly, Eqns. 1-4 are designed to consider not only the gravitational attraction between Voyager and Jupiter, but also the ever-present gravitational force of the Sun. In this model, the Sun is given a mass value of 1. In Eqns. 1 and 2, Jupiter's mass is defined as .001, or 1/1000 that of the Sun. This value computes the force felt by Voyager 2 as the distance between it and Jupiter decreases.

Secondly, Jupiter and Voyager 2 must cross orbits for any gravitational interaction to occur. The model must establish a set orbit for Jupiter, so that Voyager can accurately intercept the Jovian gravitational field. Taking Jupiter's average orbital radius as 5.2 AU, a given equation can be used to measure the Jovian orbital period, assuming a simple circular orbit:

$$P_j = 2\pi(5.2)^{3/2}(1+m_j)^{-1/2}. \tag{5}$$

Eqns. 5 contains the variable m_j in order to determine orbital period based upon Sun-Jupiter gravitational attraction, in accordance with Newton's definition of gravity:

$$F_g = \frac{GM_{sun}M_{jupiter}}{r^2}. \tag{6}$$

Furthermore, from the orbital period of Eqns. 5, the coordinates of Jupiter on any given day can be ascertained. The x_j and y_j points contained within Eqns. 1 and 2, as accurate functions of time, are

$$x_j = -3.535\cos\frac{1}{5.2^{3/2}(1+m_j)^{-1/2}(t-t_0)} \tag{7}$$

$$y_j = 3.956\sin\frac{1}{5.2^{3/2}(1+m_j)^{-1/2}(t-t_0)}. \tag{8}$$

Here, the the leading coefficients represent the initial location of Jupiter in the developed coordinate plane at t_0, the starting time for approximation.

By defining m_j, x_j, and y_j, along with the orbital radius of Jupiter (Eqns. 3), the distance between Jupiter and Voyager 2 (Eqns. 4), and initial conditions for location of both orbiting bodies, the Runge-Kutta program has the information needed to produce an accurate model for the trajectory of Voyager 2 as it intercepts Jupiter on its journey outward from the Sun.

IV. RUNGE-KUTTA CONVERSIONS

Recall the master differential equations for modeling Voyager's trajectory:

$$x'' = \frac{-x}{r^3} - \frac{m_j(x-x_j)}{D^3}$$

$$y'' = \frac{-y}{r^3} - \frac{m_j(y-y_j)}{D^3}.$$

Now, while all the variables have been established, the Runge-Kutta approximation program in Mathematica cannot produce results with two second-order differential equations. To sensibly compute results, Eqns. 1 and 2 must be converted into four separate, first-order equations. Using x as an arbitrary program variable, we assume:

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$x_3 = y$$

$$x_4 = \dot{y}.$$

By differentiating each x-form, we arrive at:

$$\dot{x_1} = \dot{x} = x_2$$

$$\dot{x_2} = \ddot{x} = \frac{-x}{r^3} - \frac{m_j(x-x_j)}{D^3}$$

$$\dot{x_3} = \dot{y} = x_4$$

$$\dot{x_4} = \ddot{y} = \frac{-y}{r^3} - \frac{m_j(y - y_j)}{D^3}.$$

Two second-order equations are now expanded to four first-order equations, all of which contain a form of a single variable, x. Essentially, the original equations are contained within x_2 and x_4, while x_1 and x_3 help to define the coordinate variables, x and y.

Mathematica can now interpret these differential equations using a Runge-Kutta approximation for a system of four first-order equations, and the final variable required for computation is time. The program is set to evaluate from t_0 to t_4, to show the four month period from May 1, 1979, to September 1, 1979, in which Voyager 2 was under the influence of the Jovian gravitational field.

V. GRAPHICAL RESULTS

When allowed to run with a step size of $h = .01$, the Runge-Kutta approximation produces a striking plot of Voyager 2's path around Jupiter (Appendix 2)

The plot originates at Voyager's initial celestial y-coordinate, 3.612 AU, and follows its trajectory as it tracks around Jupiter. This graph plots Voyager's x and y coordinates against each other, producing a map of the satellite's path. The visual image is a very clear "slingshot" pattern around the gas giant. The second y-axis intercept occurs at about -10 AU, showing that Voyager 2 has been flung 6.388 AU farther away from the sun than when measurements began. Clearly, Jupiter has greatly influenced the travelling probe. Jupiter's maximum x-coordinate displacement is very close to its initial x_j position, and we see the most gravitational bending at this point. This fact confirms a close encounter between Voyager and Jupiter, so that Voyager received a significant trajectory alteration. Without Jupiter's manipulation, Voyager could have never continued its Grand Tour.

VI. ENERGY ABSORPTION

In addition to the needed trajectory change, Voyager 2 had to absorb orbital energy from Jupiter in order to continue its mission. Most of Voyager's original launch velocity (36 km/s), had bled off by the time the probe reached the outer planets, courtesy of our massively attractive star. Without a quantitative boost of velocity from Jupiter, Voyager 2 would have gently slowed and then tumbled back towards the Sun. Buried within the trajectory map, data exists concerning the instantaneous velocity of Voyager. Because both Jupiter's orbital period and distance (the components of velocity) are known, we can use measurement of Voyager's closing distance upon Jupiter to deduce how much energy the probe steals from the planet. A handy Mathematica code extracts this data into a plot of day-by-day velocity for Voyager. The resulting graph is just as striking as the trajectory map (Appendix 3).

On this plot, we see Voyager rapidly gaining speed as it approaches Jupiter. Because the distance between the Sun and Voyager is enormous in comparison to the encounter between the planet and the probe (remember Eqns. 6: gravitational attraction falls off as the square of the distance), Jupiter is almost solely in control of the probe's velocity and trajectory. Voyager experiences a nearly straight-in acceleration during its approach to Jupiter, resulting in a maximum velocity close to initial launch speed. The back half of the plot illustrates a subsequent deceleration experienced by Voyager 2, due to the fact that as it races away, Jupiter now pulls back on the satellite. As time passes and Voyager increases its separation from Jupiter, the gravitational attraction and deceleration wane. Thankfully for NASA mathematicians, Voyager exits the system with more energy and velocity than when entering, so it can continue on its journey. This energy was actually "stolen" from Jupiter, because as the planet pulls on the probe, the probe also pulls back. Jupiter, being the exponentially more massive object, cannot measure its energy loss, but it still exists. By sending hoards of probes past Jupiter, we could actually measure the decrease in the planet's orbital velocity, though the consequences of such actions would prove disastrous as Jupiter falls from its current, stable orbit.

VII. MODEL ACCURACY

The true test for modeling gravitational effects upon Voyager 2 comes with accuracy when comparing to real-world data. Firstly, the modeled location of Voyager's gravitational assist agrees well with actual celestial data concerning Jupiter's location, as previously stated. Location discrepancies of $\pm .05AU$ occur because solar latitude was disregarded when formulating the model. Over the indescribably vast distances of space, even small angles of inclination translate into large distances when extended out far enough. A three-dimensional model involving multiple angles of celestial inclination provides for a great challenge, but would not have agreed with the differential equations used in the model. The two-dimension celestial plane allowed for a solvable study.

Secondly, the velocity gained by Voyager on its trip past Jupiter agrees extremely well with actual graphs from NASA and the Jet Propulsion Laboratory (JPL) (Appendix 4).

The features of both plots are very similar. Concentrating on the function in the vicinity of Jupiter, we see a sharp acceleration followed by a partial deceleration. Even without exact values, the deceleration portion of the graphs provide convincing evidence that the model is a good interpretation of actual events. On the JPL graph, notice the deceleration after Jupiter, and how it compares to the decelerations after Saturn, Uranus, and Neptune. As a result of its much greater gravity, Jupiter creates a much broader deceleration curve than the other outer planets. The velocity plot created from modeled data mimics this tendency very well. Even though both exhibit evident decelerations, they clearly show that a net increase in velocity was the final result. Again, without this quantitative boost in speed, Voyager could have never extended its mission into interstellar space.

Finally, the actual trajectory model of Voyager, while convincing of the "slingshot" encounter, does vary from NASA data. The modeled path around Jupiter shows a trajectory that is slightly sharper than expected. The actual angle of deflection from the tangent of the hyperbolic path should be 53°, but the developed plot has an angle closer to 60°. The accuracy of the plot could be altered by several factors. First, the system presented is a closed Sun-Jupiter-Voyager construct. Obviously, actual solar space is populated by countless bodies that interacted with the Voyager probe. Minute interactions with other gravitational fields, those not considered in this model, can, with time, have slight impacts upon Voyager's path through the solar system. Secondly, and most importantly, Voyager 2's approach angle towards Jupiter was altered when the solar latitudes were condensed. By constraining the gravitational encounter to only two dimensions, variances from actual data arose.

VIII. ADDITIONAL ANALYSIS AND CONCLUSION

Undoubtedly, modeling Voyager 2's trajectory past Jupiter can be expanded to encompass its entire journey through the solar system. This model could be redeveloped to next encounter Saturn's gravitational field, then Uranus's, and finally Neptune's. The total variance between model and actual data upon leaving the planetary solar system would prove to be an interesting conclusion of the expanded version. Because the elimination of solar latitude proved to cause some discrepancies, a more developed model could contain correctors for the differing angles of inclination, though more complicated equations may be needed and the danger of performing actual rocket science becomes apparent. This model, though a good approximation of gravitational assistance, pales to the exacting work of the mathematicians, engineers, and physicists that made the Voyager 2 program such a resounding success. Their ability to plan a trajectory that relies upon the both the great power and exacting delicacy of gravitational fields throughout the solar systems has resulted in the discovery of hundreds of new solar objects, as well as worthwhile data that will take many years to study. The Grand Tour of Voyager 2 will be remembered as one of the greatest applications of math and science in the modern space age.