Betsy Takes A Lunar Path

Paul Kolmodin and Amos Sookraj

Abstract: Given the set of differential equations governing the motion of three bodies in space, we can find a set of initial conditions that will successfully send a cow over the moon and back to Earth.

Introduction

Hey diddle, diddle
The Cat and the Fiddle
The Cow jumped over the Moon
...

Our fascination with the moon took a new turn when we were reintroduced to the nursery rhyme Hey Diddle, Diddle[1]. We wondered if we could use a mathematical model to simulate a cow jumping over the Moon. We begin this paper by finding differential equations that describe the orbit of a satellite around a large celestial body. Ignoring the effect of the Earth upon the Moon, we find initial conditions that simulate the cow jumping over the moon. Throughout this project, the mass of the cow, while significant to the cow, is assumed to be insignificant to the project at hand.

Equations of Motion

We use the following equations in our modeling project.[2]

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[1]Whitaker, p61
Whitaker, H.C. 1997. Problems from 100 years Ago in the Monthly... *The American Mathematical Monthly* Vol 104-1.

[2]Danby, p115-117
Danby, J.M.A. 1985. Computing Applications to Differential Equations. Reston Va. Reston Publishing Company Inc.

$$
\begin{aligned}
r &= \left(x^2 + y^2\right)^{1/2} \\
D &= \left[\left(x - x_m\right)^2 + \left(y - y_m\right)^2\right]^{1/2} \\
\ddot{x} &= \frac{-x}{r^3} - \frac{M_m\left(x - x_m\right)}{D^3} \\
\ddot{y} &= \frac{-y}{r^3} - \frac{M_m\left(y - y_m\right)}{D^3}
\end{aligned}
$$

The first term of the differential equation for $\ddot{x}$, $-x/r^3$, accounts for the Earth's gravitational pull on a body in the x direction. The second portion of this equation $-M_m\left(x - x_m\right)/D^3$ accounts for the moon's gravitational pull on a body in the x direction. Because of the nature of gravity, the effects of a planet's gravitational field on the cow will only be significant when the cow is in proximity to the planet. We used D to represent the distance that the cow is from the center of the Moon. Also, r signifies the distance between the cow and the Earth.

Likewise the first term of the differential equation for $\ddot{y}$, $-y/r^3$, accounts for the Earth's gravitational pull on a body in the y direction. The second portion of this equation $-M_m\left(y - y_m\right)/D^3$ accounts for the moon's gravitational pull on a body in the y direction.

Initial Conditions, Unit of Measurements and First Attempt

We start by assigning the center of the Earth as (0,0) of the Cartesian plane. From there we place the Moon in a circular orbit of radius 1 around the Earth. To determine the scale of the graph we need to know some important facts, listed below.

radius of the Earth = 6764 kilometers
radius of the Moon = 1691 kilometers
mass of the Moon = 0.01230002 w / respect to mass of Earth
distance to the Moon = 384400 kilometers from center of Earth to center of Moon
full orbit of the Moon = 29 days
Betsy = the cow which will jump over the Moon, and safely return to Earth.

A distance of 1 on the Cartesian plane shall be defined as 384400 kilometers. We define an object with a distance of 6764 / 384400 project distance units from the origin as standing on the Earth's surface. A speed of 10 in our project units equals the escape velocity of 40320 kilometers per hour[3].

Once we get the Moon orbiting the Earth, we start our Betsy at the point (0.1, 0) which we assume as being in orbital status and escaped from the Earth's atmosphere. From there we give Betsy an initial velocity vector of <4.3,0> project velocity units, which equals a velocity of 17337.6 kilometers per hour. We find that this initial velocity sends Betsy around the Moon and on course for Earth. The initial position of the Moon lies in the fourth quadrant of the Cartesian plane represented by a point. See **Figure 1**.

We turn to *Mathematica* ™ and using the Runge-Kutta[4] numerical approximation method for four first order equations; we continue the process of finding Betsy's lunar path. The Runge-Kutta method is a numerical approximation method that builds on Simpson's Rule. Simpson's Rule uses parabolas to approximate area under a curve.

The Runge-Kutta method needs a few initial conditions to get Betsy on the lunar path. We decide to look 2.8 project time units into the future, or 17.48572 hours and we use a step size of 0.0014.

Figure 2-a illustrates Betsy on her lunar path. **Figure 2-b** and **Figure 2-c** demonstrate Betsy crossing the Moon's orbital path ahead of the Moon. **Figure 2-d** shows Betsy reaching her highest peak. Next Betsy begins her trip homeward, as shown in **Figure 2-e**. Finally Betsy is on her way to her original orbital state because she is now in the Earth's gravitational pull, see **Figure 2-f**.

Final Results

Our next simulation involves placing Betsy on the surface of the Earth by changing Betsy's initial position from (0.1, 0) to (0.0175962539, 0). We obtain 0.0175962539 by taking the radius of the Earth divided by the distance to the Moon. We will look 1.75 units into the future, or 10.928575 hours and use a step size of 0.00075. We find Betsy's initial velocity vector to be <10.577,0>, which equals a velocity of 42646.464 kilometers per hour. We then simulate Betsy's lunar path and find success! The next paragraph is a description of Betsy's voyage.

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[3]Danby, p116

[4]Schaum's Outline p 254-271
Bronson, Richard. 1973. Schaum's Outline of Theory and Problems of Differential Equations. 2nd Edition. USA. Mc-Graw Hill

Betsy starts off on the Earth's surface, see **Figure 3-a**. **Figures 3-b** and **3-c** shows Betsy beginning her ascension to the Moon's orbital path. Betsy approaches the Moon's path in **Figure 3-d** and **3-f** while the Moon advances. At this juncture in time, Betsy jumps over the Moon, as in **Figure 3-g**, **3-h** and **3-i**. Finally **Figure 3-j**, **3-k**, and **3-l** show Betsy heading for home.

Conclusion

Figures 3-a-l, may show that Betsy jumped over the Moon, but there is always the possibility that Betsy crashed into the Moon. To prove that Betsy never crashed into the Moon's surface, we make a graph of the position of Betsy compared to the Moon's center.

When one looks closely at **Figure 4**, one observes that Betsy never comes closer than 0.015 project distance units, which is approximately 0.015 * 384400, or 5766 km from the center of the Moon. We find the Moon's radius to be 1691 km. This means that Betsy never comes closer than 4075 kilometers from the Moon's surface and that Betsy has successfully jumped over the Moon.

Finally Betsy arrives on the Earth with a velocity of 30855.924 kilometers per hour. We confirm touchdown at point (0.00990115, 0.0122794).

...

The Little Dog laughed to see such Sport
And the Dish ran away with the Spoon.

Directions For Further Research

The next step is to test the sensitivity of the system to varying initial conditions, such as initial velocity, or initial position of the Moon. Duplicating the success in two dimensions, one could expand this project into three dimensional space. Also, we hope to include the effects on Betsy from the other objects in the Solar System.

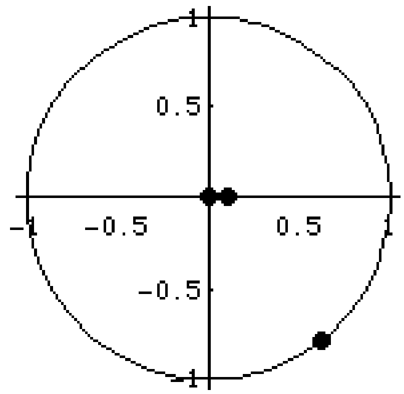


Figure 1

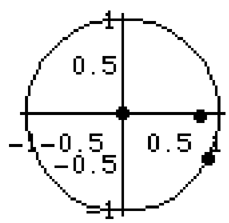


Figure 2-a

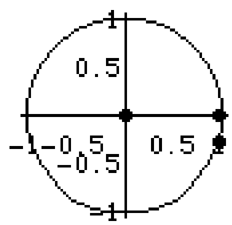


Figure 2-b

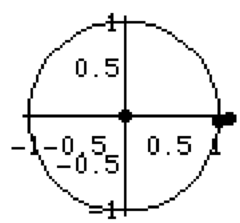


Figure 2-c

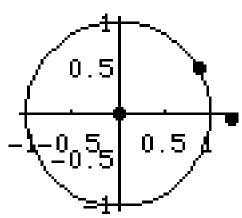


Figure 2-d

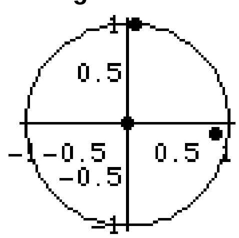


Figure 2-e

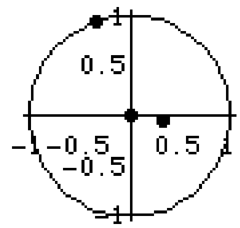


Figure 2-f

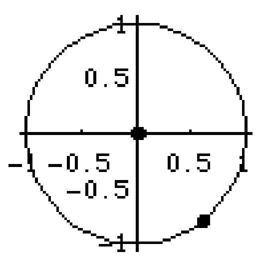


Figure 3-a

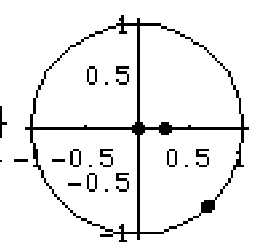


Figure 3-b

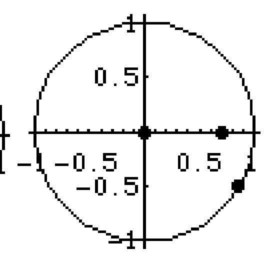


Figure 3-c

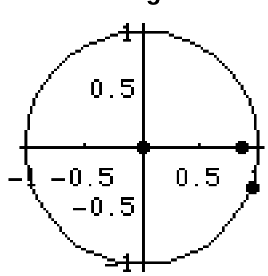


Figure 3-d

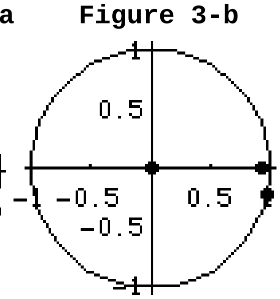


Figure 3-e

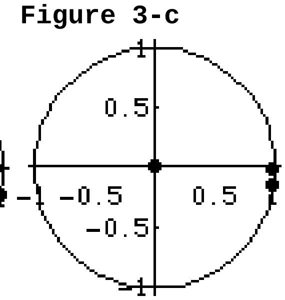


Figure 3-f

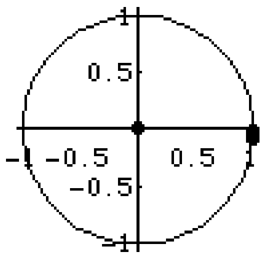


Figure 3-g

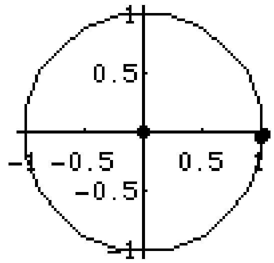


Figure 3-h

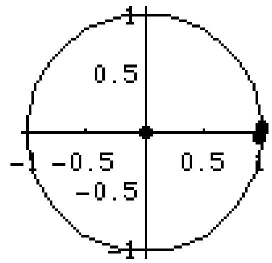
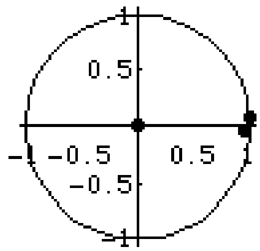


Figure 3-i



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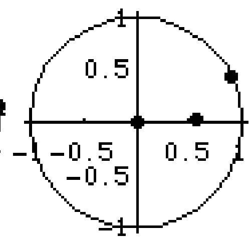

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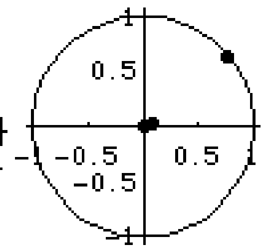

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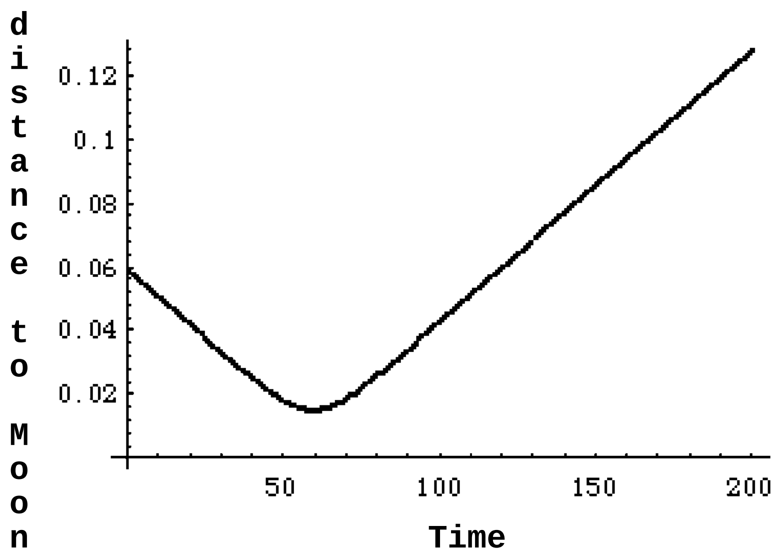

Figure 4